Scalar Triple Product

- $\vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}]$ is a scalar quantity.
- $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{b} \cdot (\vec{c} \times \vec{a})$
  $= \vec{c} \cdot (\vec{a} \times \vec{b}) = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$

That means dot and cross can be interchanged.
Also, the position of $\vec{a}, \vec{b}, \vec{c}$ can be changed as $\vec{a}$ to $\vec{b}$, $\vec{b}$ to $\vec{c}$, $\vec{c}$ to $\vec{a}$

- $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}]$
- If two vectors are identical, then the scalar triple product $= 0$
  i.e. $[\vec{a}\ \vec{a}\ \vec{b}] = 0 = [\vec{a}\ \vec{b}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{b}]$
- $[\vec{i}\ \vec{j}\ \vec{k}] = 1$
- $[\vec{a}\ \vec{a}\ \vec{a}] = 0$
- If the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then
  $$[\vec{a}\ \vec{b}\ \vec{c}] = 0$$
- If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$
  $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$, then

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Ex 1. Find $(2\vec{i}+3\vec{j}-5\vec{k})\times(3\vec{i}-2\vec{j}+6\vec{k})\cdot(5\vec{i}+\vec{j}+\vec{k})$

Soln. The given expression $= \begin{vmatrix} 2 & 3 & -5 \\ 3 & -2 & 6 \\ 5 & 1 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 5 & 1 & 1 \\ 3 & -2 & 6 \\ 5 & 1 & 1 \end{vmatrix} \quad \text{By } R_1 \to R_1 + R_2$$

$$= 0$$

Ex 2. Find $(\vec{a}\times\vec{b})\cdot\vec{c}$ where $\vec{a} = 2\vec{i}-3\vec{j}-\vec{k}$, $\vec{b} = 2\vec{i}+\vec{j}-\vec{k}$, $\vec{c} = \vec{i}-\vec{j}+2\vec{k}$.

Soln. $\vec{a}\times\vec{b} = (2\vec{i}-3\vec{j}-\vec{k})\times(2\vec{i}+\vec{j}-\vec{k})$

$$= -6\,\vec{j}\times\vec{i} - 2\,\vec{k}\times\vec{i} + 2\,\vec{i}\times\vec{j} - \vec{k}\times\vec{j} - 2\,\vec{i}\times\vec{k} + 3\,\vec{j}\times\vec{k}$$

$$= 6\vec{k} \cancel{-2\vec{j}} + 2\vec{k} + \vec{i} \cancel{+2\vec{j}} + 3\vec{i}$$

$$= 4\vec{i} + 8\vec{k}$$

$\therefore (\vec{a}\times\vec{b})\cdot\vec{c} = (4\vec{i}+8\vec{k})\cdot(\vec{i}-\vec{j}+2\vec{k})$

$$= 4\times1 - 0 + 8\times2 = 20$$

- Volume of a parallelepiped whose edges are $\vec{a}, \vec{b}, \vec{c}$
$= \vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}]$

Q. Find the volume of a parallelepiped whose edges are $\vec{i}+2\vec{j}+3\vec{k}$, $3\vec{i}+7\vec{j}-4\vec{k}$ and $\vec{i}-5\vec{j}+3\vec{k}$.

Soln. The volume of a parallelepiped

$$= (\vec{i}+2\vec{j}+3\vec{k})\cdot(3\vec{i}+7\vec{j}-4\vec{k})\times(\vec{i}-5\vec{j}+3\vec{k})$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 7 & -4 \\ 1 & -5 & 3 \end{vmatrix} \quad \text{By } R_1 \to R_1 - R_3$$

$$= \begin{vmatrix} 0 & 7 & 0 \\ 3 & 7 & -4 \\ 1 & -5 & 3 \end{vmatrix} = -7\begin{vmatrix} 3 & -4 \\ 1 & 3 \end{vmatrix} = -7(9+4) = -91$$

Since, volume cannot be negative, so the required volume $= 91$ cubic unit.